EXPANSIONS OF $B_m^{(n)}$ AND a_n INTO POWER-SERIES WITH RESPECT TO τ.

R 53, Int 5.

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INTRODUCTION

This report R 53, Int 5 is the fifth of a number of interim reports giving information about computations carried out by the Computation Department of the Mathematical Centre on behalf of the National Aeronautical Research Institute in Amsterdam under contract R 53. The final report R 53 that will be made up eventually will not contain much else than the final results of the computations and, moreover, will be not available for general distribution. As however in the course of the computations a lot of information has to be compiled for internal use, and part of this information may be of some value to others, this compilation will be done in the form of interim reports, that will be made available for limited circulation.

Expansions of $B_m^{(n)}$ and a_n into power-series with respect to τ.

For small τ we can expand the Fourier-coefficients $B_m^{(n)}$ and the characteristic values a_n into power-series with respect to τ. Some results may be found for instance in Mac.Lachlan "Theory and application of Mathieu Functions".

1. The case n = 1.

First we treat the case n = 1 :

We put

$$\left.\begin{aligned} b_1 &= 1 + \sum_{h=1}^{\infty} \alpha_h \tau^h \\ B_{2r+1}^{(1)} &= \sum_{h=r}^{\infty} \beta_h^{(r)} \tau^h \end{aligned}\right\} \qquad (1,1)$$

$$(\tau = q/4)$$

We make use of the recurrence relations:

$$\begin{aligned} &(b_1 - 1 + q)\, B_1^{(1)} - q\, B_3^{(1)} = 0 \\ &\left[b_1 - (2r+1)^2\right] B_{2r+1}^{(1)} - q \left\{ B_{2r+3}^{(1)} + B_{2r-1}^{(1)} \right\} = 0 \end{aligned} \qquad (1,2)$$

The last equation yields the relations:

$$- r(r+1)\, \beta_r^{(r)} = \beta_{r-1}^{(r+1)}$$

$$- r(r+1)\, \beta_{r+1}^{(r)} = \beta_r^{(r-1)} - \alpha_1 \beta_r^{(r)}/4$$

$$- r(r+1)\, \beta_{r+\nu}^{(r)} = \beta_{r+\nu-1}^{(r-1)} - 1/4 \sum_{h=1}^{\nu} \alpha_h \beta_{r+\nu-h}^{(r)} + \beta_{r+\nu-1}^{(r+1)} \{\nu > 1\}$$

The calculation becomes somewhat easier by putting

$$\beta_{r+\nu}^{(r)} = \frac{(-1)^r}{r!\,(r+1)!}\, c_\nu^{(r)} \qquad (1,3)$$

The first formula of (1,2) then changes into:

$$4\tau + \sum_{h=1}^{\infty} \alpha_h \tau^h = 4\tau \sum_{h=1}^{\infty} \beta_h^{(1)} \tau^h$$

or

$$\left.\begin{aligned} \alpha_1 &= -4 \\ \alpha_h &= 4\beta_{h-1}^{(1)} \qquad h > 1 \end{aligned}\right\}. \qquad (1,4)$$

So we find successively:

$$\beta^{(r)}_{r} = \frac{(-1)^r}{r!\,(r+1)!} ,$$

$$\beta^{(r)}_{r+1} = \frac{(-1)^{r+1}}{[(r+1)!]^2} ,$$

$$\beta^{(r)}_{r+2} = \frac{(-1)^r}{4(r-1)!\,(r+2)!} ,$$

$$\beta^{(r)}_{r+3} = \frac{(-1)^r \{3r+8\}}{8(r-1)!\,(r+2)!\,(r+2)} ,$$

$$\beta^{(r)}_{r+4} = \frac{(-1)^{r+1}\{67r^2+355r+460\}}{144(r-1)!\,(r+3)!\,(r+2)} ,$$

$$\beta^{(r)}_{r+5} = \frac{(-1)^r\{21r^3+166r+432r+371\}}{144(r-1)!\,(r+3)!\,(r+2)(r+3)} ,$$

$$\beta^{(r)}_{r+6} = \frac{(-1)^r\{1427r^4+17686r^3+81019r^2+162408r+119924\}}{6912(r-1)!\,(r+4)!\,(r+2)(r+3)} .$$

We have therefore in particular

$$b_1 = 1 - 4\tau - 2\tau^2 + \tau^3 - \frac{1}{6}\tau^4 - \frac{11}{36}\tau^5 + \frac{49}{144}\tau^6 - \frac{55}{576}\tau^7 - \frac{83}{540}\tau^8 \ldots \qquad (1,5)$$

The calculation of $T^{(1)}_{2r+1}$ and $g^{(1)}_{2r+1}$, defined in the report R 53 Int 2,2 (9.2) and (9,1) is now easy:

$$\left.\begin{aligned} T^{(1)}_{2r+1} &= -\frac{(-1)^r}{2}\,\tau^r \frac{d}{dr}(-1)^r \tau^{-r} B^{(1)}_{2r+1} , \\ B^{(1)}_{2r+1} &= \tau^r \sum_{h=0}^{\infty} \beta^{(r)}_{h+2}\,\tau^h = \frac{(-\tau)^r}{r!\,(r+1)!} \sum_{h=0}^{\infty} c^{(r)}_h \tau^h , \\ g^{(1)}_{2r+1} &= T^{(1)}_{2r+1} + \vartheta B^{(1)}_{2r+1} . \end{aligned}\right\} \qquad (1,6)$$

First we compute $T^{(1)}_{2r+1}$:

$$\begin{aligned} T^{(1)}_{2r+1} &= -\frac{(-\tau)^r}{2}\frac{d}{dr}\sum_{h=0}^{\infty}\frac{c^{(r)}_h \tau^h}{r!\,(r+1)!} \\ &= \frac{(-\tau)^r}{2}\,\frac{\psi(r)+\psi(r+1)}{r!\,(r+1)!}\sum_{h=0}^{\infty} c^{(r)}_h \tau^h - \frac{(-\tau)^r}{2r!\,(r+1)!}\sum_{h=0}^{\infty}\frac{dc^{(r)}_h}{dr}\tau^h \\ &= \frac{\psi(r)+\psi(r+1)}{2} B^{(1)}_{2r+1} - W_{2r+1} , \end{aligned} \qquad (1,7)$$

with $$W_{2r+1} = \frac{(-\tau)^r}{2r!\,(r+1)!} \sum_{h=0}^{\infty} \frac{dc_h^{(r)}}{dr}\,\tau^h \qquad (1,8)$$

and the Ψ-function defined as the logarithmic derivative of the factorial-function.

To compute ϑ we use the recurrence formula for the $g_m^{(n)}$

$$(b_1 - 1 - q)\, g_1^{(1)} - q\, g_3^{(1)} = -2\, B_1^{(1)}. \qquad (1,9)$$

Substitution of the last equation of (1,6) gives

$$\vartheta = \frac{1}{q} + \frac{(b_1-1-q)T_1^{(1)} - q\, T_3^{(1)}}{2\, q\, B_1^{(1)}} =$$

$$= \frac{1}{q} - \frac{3(b_1-1)}{8\, q} + C - \frac{7}{8} - \frac{(b_1-1-q)}{2\, q}\, W_1 + \frac{W_3}{2}, \qquad (1,10)$$

with $C = -\Psi(0) = -\left\{ \frac{d}{dx} (\log x!) \right\}_{x=0}$.

So the result is:

$$W_{2r+1} = \frac{(-\tau)^r}{r!\,(r+1)!} \Bigg[-\frac{\tau}{2(r+1)^2} + \frac{\tau^2}{4(r+2)^2} + \frac{\tau^3(r+4)}{4(r+2)^3} - \frac{(19r^3+166r^2+480r+460)\tau^4}{48(r+2)^3\,(r+3)^2}$$

$$+ \frac{44r^4+470r^3+1875r^2+3329r+2226}{288\,(r+2)^3(r+3)^3}\,\tau^5$$

$$+ \frac{1146r^6+23130r^5+192864r^4+849732r^3+2084744r^2+2698552r+1439088}{6912\,(r+2)^3\,(r+3)^3\,(r+4)}\,\tau^6 \ldots \Bigg] \qquad (1,11)$$

and

$$\vartheta = \frac{1}{4\tau} + C - \frac{1}{2} - \frac{5}{16}\tau - \frac{\tau^2}{8} + \frac{61}{288}\tau^3 - \frac{89}{864}\tau^4 - \frac{4085}{82944}\tau^5$$

$$+ \frac{148193}{497664}\tau^6 \ldots. \qquad (1,12)$$

<u>Note</u>: We have put in all formula's $B_1^{(1)} = 1$ and so we still have to normalise by means of the relation:

$$\sum_{m=1}^{\infty} \left\{ B_m^{(1)} \right\}^2 = 1 \qquad (1,13)$$

2. <u>The case n = 2.</u>

By putting
$$b_2 = 4 + \sum_{h=1}^{\infty} \alpha_h \tau^h$$

$$B_{2r}^{(2)} = \sum_{h=r-1} \beta_h^{(r)} \tau^h \qquad (2,1)$$

and making use of the recurrencerelations:

$$\left.\begin{aligned} &(b_2 - 4)\, B_2^{(2)} - q\, B_4^{(2)} = 0 \\ &(b_2 - 4r^2)\, B_{2r}^{(2)} - q(B_{2r+2}^{(2)} + B_{2r-2}^{(2)}) = 0 \qquad r \geqq 2 \end{aligned}\right\}, \qquad (2,2)$$

we get

$$\left\{- 4(r-1)(r+1) + \sum_{h=2}^{\infty} \alpha_h \tau^h\right\} \sum_{h=r-1}^{\infty} \beta_h^{(r)} \tau^h = 4\tau\left\{\sum_{h=r}^{\infty} \beta_h^{(r+1)} \tau^h + \right.$$
$$\left. + \sum_{h=r-2}^{\infty} \beta_h^{(r-1)} \tau^h\right\}$$

from which it follows that:

$$-(r-1)(r+1)\,\beta_{r-1}^{(r)} = \beta_{r-2}^{(r-1)}$$

$$-(r-1)(r+1)\,\beta_r^{(r)} = \beta_{r-1}^{(r-1)}$$

$$-4(r-1)(r+1)\,\beta_{r+\nu}^{(r)} = 4\beta_{r-1+\nu}^{(r-1)} - \sum_{h=2}^{\nu+1} \alpha_h \beta_{r+\nu-h}^{(r)} + 4\beta_{r-1+\nu}^{(r+1)} \quad (\nu \geqq 1)$$

Again the calculation becomes somewhat easier by putting:

$$\beta_{r+\nu}^{(r)} = \frac{(-1)^{r-1}}{(r-1)!\,(r+1)!}\, c_\nu^{(r)} \qquad (2,3)$$

Further we know $B_0^{(2)} = 0$ and we put $B_2^{(2)} = 1$. After all we have also to normalise by means of the relation (1,13) now applied for the case n = 2.

The first formula of (2,2) yields

$$\sum_{h=1}^{\infty} \alpha_h \tau^h = 4\tau \sum_{h=1}^{\infty} \beta_h^{(2)} \tau^h$$

or

$$\left.\begin{aligned} &\alpha_1 = 0 \\ &\alpha_h = 4\beta_{h-1}^{(2)} \qquad h > 1 \end{aligned}\right\} \qquad (2,4)$$

The results of the computations are:

$$\beta_{r-1}^{(r)} = \frac{2(-1)^{r-1}}{(r-1)!\,(r+1)!},$$

$$\beta_{r+1}^{(r)} = \frac{(-1)^r(7r+16)}{18(r-2)!(r+2)!(r+1)} ,$$

$$\beta_{r+3}^{(r)} = \frac{(-1)^{r-1}(197r^2+1120r+1596)}{2592(r-2)!(r+2)!(r+2)(r+3)} ,$$

$$\beta_{r+5}^{(r)} = \frac{(-1)^r(77077r^4+961275r^3+4438124r^2+8980500r+6701472)}{46656\ (r-2)!(r+1)!(r+2)^2(r+3)^2(r+4)} ,$$

$$\beta_{r+2}^{(r)} = \beta_{r+4}^{(r)} = \beta_{r+6}^{(r)} = 0,$$

$$b_2 = 4 - \frac{4}{3}\tau^2 + \frac{5}{54}\tau^4 - \frac{289}{19440}\tau^6\ \frac{21391}{6998400}\tau^8 \ldots. \tag{2,5}$$

The calculations of $T_{2r}^{(2)}$, $g_{2r}^{(2)}$ and ϑ are similar to those in the case n = 1. Now we have:

$$\left.\begin{aligned} T_{2r}^{(2)} &= -\tfrac{1}{2}(-\tau)^r \frac{d}{dr}(-\tau)^{-r} B_{2r}^{(2)} , \\ g_{2r}^{(2)} &= T_{2r}^{(2)} + \vartheta B_{2r}^{(2)} \\ B_{2r}^{(2)} &= \sum_{h=r-1}^{\infty} \beta_h^{(r)} \tau^h = \sum_{h=0}^{\infty} (-\tau)^{r-1} \frac{c_{h-1}^{(r)}\ \tau^h}{(r-1)!(r+1)!} . \end{aligned}\right\} \tag{2,6}$$

Again we have:

$$T_{2r}^{(2)} = \frac{\Psi(r-1)+\Psi(r+1)}{2} B_{2r}^{(2)} - W_{2r}, \tag{2,7}$$

$$\text{with } W_{2r} = \frac{(-\tau)^{r-1}}{2} \frac{\sum_{\nu=-1}^{\infty} \frac{\partial c_\nu^{(r)}}{\partial r} q^{\nu+1}}{(r-1)!(r+1)!} \tag{2,8}$$

So we find:

$$T_{2r}^{(2)} + W_{2r}^{(2)} = \frac{\Psi(r-1)+\Psi(r+1)}{2} B_{2r}^{(2)} =$$

$$= (-\tau)^{r-1} \sum_{\nu=-1}^{\infty} c_\nu^{(r)}\ q^{\nu+1} \left\{ \frac{-1}{2\{(r+1)!\}^2} + \frac{\Psi(r)}{(r-1)!(r+1)!} \right\} .$$

r = 0 gives:

$$T_0^{(2)} + W_0 = \frac{2}{q} \sum_{\nu=-1}^{\infty} c_\nu^{(0)}\ q^{\nu+1}$$

and by putting $W_0 = 0$, $B_0^{(2)} = 0$, $B_2^{(2)} = 1$ we find the following expression for ϑ

$$\vartheta = \frac{b_2}{2q} T_0^{(2)} - T_2^{(2)} = \frac{b_2}{q^2} \sum_{\nu=-1}^{\infty} c_\nu^{(0)} q^{\nu+1} - \frac{\Psi(0)+\Psi(2)}{2} + W_2$$

Our final expressions are:

$$W_{2r} = \frac{(-\tau)^{r-1}}{(r-1)!(r+1)!} \left\{ -\frac{2r^2+10r+11}{6(r+1)^2(r+2)^2} \tau^2 + \frac{38r^3+370r^2+1203r+1302}{432(r+2)^3(r+3)^2}\tau^4 + \right.$$

$$\left. + \frac{1624r^6+32848r^5+274985r^4+1218395r^3+3010278r^2+3928002r+2113092}{77760(r+2)^3(r+3)^3(r+4)^2}\tau^6 \ldots \right\} \quad (2,9)$$

and

$$\vartheta = C - \tfrac{3}{4} + \frac{b_2}{2}\left\{ \frac{1}{4\tau^2} + \frac{1}{18} - \frac{133}{20736}\tau^2 + \frac{23269}{74649600}\tau^4 - \right.$$

$$\left. - \frac{1906627}{13436928000}\tau^6 \ldots \right\}$$

$$- \frac{23}{432}\tau^2 + \frac{971}{124416}\tau^4 - \frac{440801}{279936000}\tau^6 \ldots \quad (2,10)$$

3. Expansions of $B_m^{(n)}$ and a_n in q series for $n \geq 6$.

In the following lines we shall omit the index $^{(n)}$, while (n) is a fixed number.

We try to write $B_m = \sum_{h=0}^{\infty} \beta_{m,h} \tau^h$ (3,1)

with the conditions

$m \neq n$ $\quad \beta_{m,h} = 0$ when $h < \frac{|n-m|}{2}$

$m = n$ $\quad B_n \equiv 1$ or $\beta_{n,0} = 1$ and $\beta_{n,h} = 0$ for $h > 0$ (3,2)

To be honest, we must say that most of the following section is only realised in the case $n = \infty$.

Don't forget the relation:

$$(b - m^2) B_m - 4\tau (B_{m-2} + B_{m+2}) = 0 \quad (3,3)$$

For b we want the expansion:

$$b = \sum_{h=0}^{\infty} \alpha_h \tau^h \text{ with } \alpha_0 = n^2. \quad (3,4)$$

Formula (3,3) combined with (3,1), (3,2) and (3,4) yields:

$$\left\{n^2-m^2 + \sum_{h=1}^{\infty} \alpha_h \tau^h\right\} \sum_{h=1}^{\infty} \beta_{m,h} \tau^h = 4 \sum_{h=1}^{\infty} \beta_{m-2,h-1} \tau^h +$$

$$+ 4 \sum_{h=1}^{\infty} \beta_{m+2,h-1} \tau^h .$$

First we eliminate α_h by treating the case m = n

$$\sum_{h=1}^{\infty} \alpha_h \tau^h = 4 \sum_{h=2}^{\infty} \beta_{n-2,h-1} \tau^h + 4 \sum_{h=2}^{\infty} \beta_{n+2,h-1} \tau^h$$

or $$\alpha_h = 4(\beta_{n-2,h-1} + {}_{n+2,h-1}) \qquad (3,4)$$

Now in the case $m \neq n$, we get:

$$\left[n^2-m^2 + 4 \sum_{h=2}^{\infty} \beta_{n-2,h-1} \tau^h + 4 \sum_{h=2}^{\infty} \beta_{n+2,h-1} \tau^h\right] \sum_{h=\frac{|n-m|}{2}}^{\infty} \beta_{m,h} \tau^h =$$

$$= 4 \sum_{h=\frac{|n-m+4|}{2}}^{\infty} \beta_{m-2,h-1} \tau^h + 4 \sum_{h=\frac{|m-n|}{2}}^{\infty} \beta_{m+2,h-1} \tau^h .$$

Equating the coëfficiënts of the p th power of τ we find successively:

m = n - 2p

$$\{n^2 -(n-2p)^2\} \beta_{n-2p,p} = 4 \beta_{n-2p+2,p-1}$$

m = n - 2p+2:

$$\{n^2 - (n-2p+2)^2\} \quad {}_{n-2p+2,p} = 4 \beta_{n-2p+4,p-1}$$

m = n - 2p + 4:

$$\{n^2 - (n - 2p + 4)^2\} \beta_{n-2p+4,p} + 4(\beta_{n-2,1} + \beta_{n+2,1}) \cdot \beta_{n-2p+4,p-2} =$$

$$= 4 \beta_{n-2p+2,p-1} + 4 \beta_{n-2p+6,p-1}$$

In all the formula's we suppose $m > 0$.

The cases $m = n + 2p$ etc. are easily computed from the cases $m = n - 2p$. We have only to replace in the factors $\{n^2-(n-2p)^2\}$, $\{n^2-(n-2p+2)^2\}$... etc. all n by -n, to get the coëfficiënts for $\beta_{n+2p,p}$, $\beta_{n+2p-2,p}$ .. etc.
When we have calculated once $\beta_{n-2p,p}$, we have to replace in the answer n by -n, and get $\beta_{n+2p,p}$.
So we find:

$$\beta_{n-2p,p} = \frac{(n-p-1)!}{p!(n-1)!} \qquad \beta_{n+2p,p} = \frac{(-1)^p n!}{p!.(n+p)!} \tag{3,5}$$

$$\beta_{n-2p+2,p} = \beta_{n+2p-2,p} = 0$$

$$\beta_{n-2p+4,p} = \frac{-(p-2)\{(n-3)p-(n+1)(n-2)\}(n-p)!}{(n+1)(n-1)^2(p-1)!(n-1)!}$$

$$\beta_{n+2p-4,p} = \frac{(-1)^p(p-2)\{(n+3)p+(n-1)(n+2)\}n!}{(n-1)(n+1)^2(p-1)!(n+p-1)!}$$

$$\beta_{n+2p-6,p} = \beta_{n-2p+6,p} = 0$$

$$\beta_{n+2p-8,p} = \frac{(-1)^p(n-3)!}{(p-4)!(n+p-4)!} \cdot C_4^{(n)}(p)$$

with

$$C_4^{(n)}(p) = \frac{1}{2(2\nu)^2(2\nu+2)^4(2\nu+3)^2}\Big\{16(12\nu^7+76\nu^6+193\nu^5+261\nu^4+197\nu^3+101\nu^2+60\nu+18)$$

$$-\frac{2\nu(2\nu+2)^4(2\nu+3)^2}{p-2} - 4\,\frac{(2\nu+2)^3(2\nu+3)^2(\nu^2-\nu+1)}{p-3} -$$

$$-\frac{(2\nu)^3(2\nu-1)(2\nu+2)^2(2\nu+3)}{n+p-2} - 8\,\frac{(2\nu-1)2\nu(2\nu+2)(2\nu+3)(\nu^3+7\nu^2+11\nu+3)}{n+p-3}\Big\}$$

where $\nu = \frac{n-1}{2}$,

and $\beta_{n-2p+8,p} = -\frac{(n-p+3)!}{(p-4)!(n+2)!} \cdot C_4^{(-n)}(p)$

where $C_4^n(p)$ is defined above, and now $\nu = \frac{-n-1}{2}$.

Further:

$$b = n^2 + \frac{8}{(n^2-1)}\tau^2 + \frac{8(5n^2+7)}{(n^2-1)^3(n^2-4)} \cdot \tau^4 -$$

$$- \frac{4(n+2)(n-3)!\,C_4^{(n)}(5)+(n-2)!\,C_4^{(-n)}(5)}{(n+2)!} \cdot \tau^6 \ldots \qquad (3,6)$$

The imitial convergence is better when n is large.

Namely: $$\frac{\beta_{n-2p,p+2}}{\beta_{n-2p,p}} \sim \frac{p}{16(p+1)}\, n^{-2} \qquad (3,7)$$

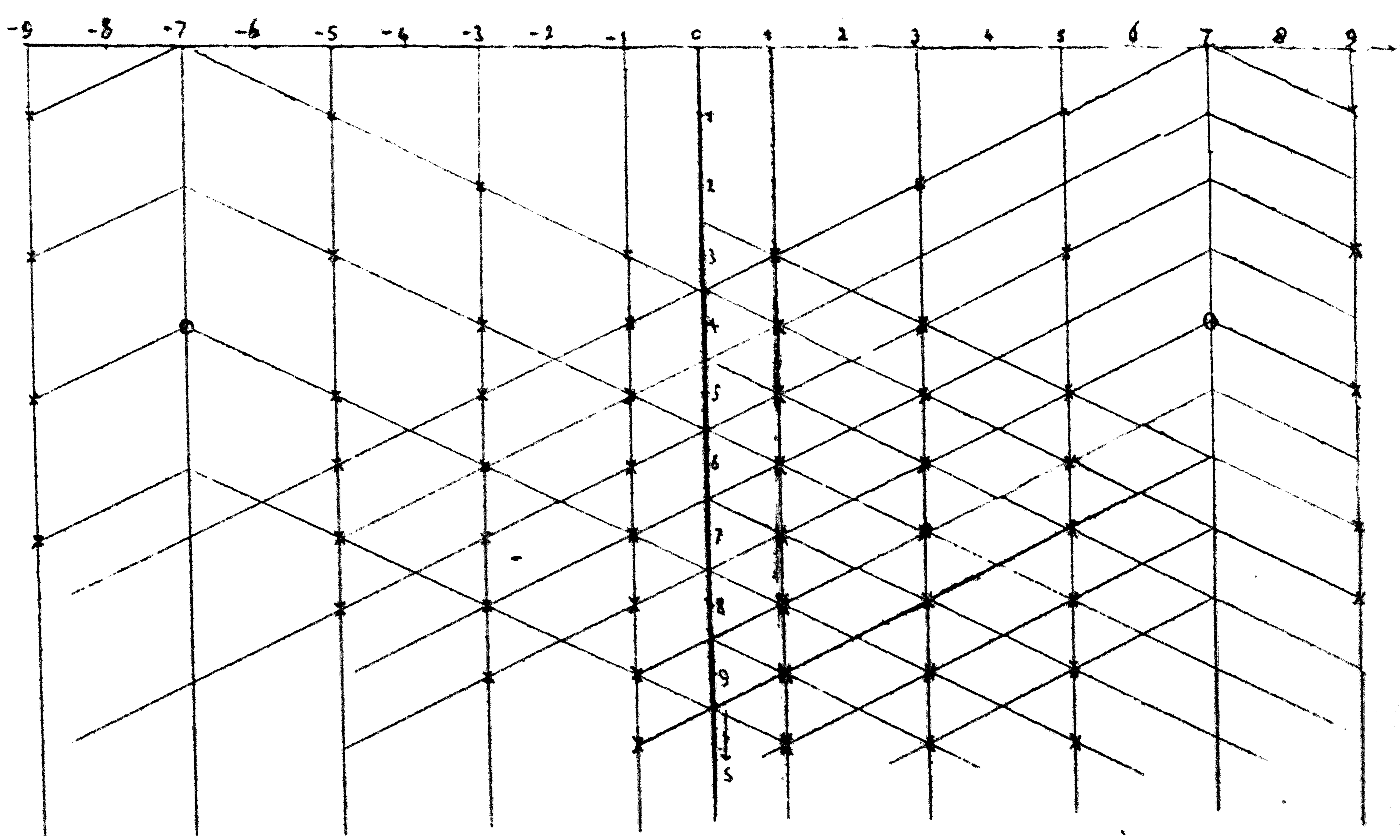
for p a constant value.

When n has a constant value and $p \to \infty$ we get:

$$\frac{\beta_{n-2p,p+2}}{\beta_{n-2p,p}} \sim \frac{n-3}{16(n+1)(n-1)^2} \qquad (3,8)$$

i.e. also a constant. For n = 6 this value is even $\frac{1}{1344}$ that is small enough for our intention

Now we have enough information to compose the scheme of the coëfficiënts $\beta_{m,s}$. Look at the figure 1. On the horizontal lines the m is pointed out, while the s is vertical. In the point (m,s) the coefficient $\beta_{m,s}$ is noted.



The cross on a place (m,s) means that the coëfficiënt $\beta_{m,s} \neq 0$. When the place is not marked $\beta_{m,s} = 0$. <u>First let n be odd:</u>
We see start sloping lines out of the not points (n,2h)(with h = 0,1,2... etc). But in the neighbourhood of the line m = 0 the situation becomes more and more complicated: We have to fulfill the antimetrical requirement for the B_m.

$$B_{-m} = - B_m . \qquad (3,9)$$

So the butterfly-figure formed by the lines starting out of (n,2h) is reflected in the line m = 0, we get also two butterfles beginning resp.: by m = 7 and m = -7.

You also see that, when n and m are large, the exponents of the expansion of $B_m^{(n)}$ are expressed by the form $\frac{|n-m|}{2} + 2s$, in the beginning, also a step 2 units large. Later this step is reduced to one unit. $B_1^{(n)}$ always has a step of one unit. Under the haevy lines our formulas (3,4) (3,5) and the results for the $\beta_{m,s}$ of the so-named "ideal case" are false. From this moment of the α_h is infected by the influence of the reflection and even $\alpha_h \neq 0$ with h odd. (See also the series-expansion for b_n in McLachlan).

When n is even, the figure is changed in this view: The sloping lines don't meet in the points (0,k + 1) with $k = \left|\frac{n}{2}\right|$, but in the points (0,k).

So the steps in the expansions rest two units large even in the coëfficiënts α_h. But the influence of the reflection forbids us again to come under the haevy lines with our computations. Therefore it is necessary to take n large. We used the formula's (3,4), (3,5) etc. only in the cases n = 6,...,10.

<u>Note</u>: One can try to write the solution of the differential-equation of the second order

$$y'' + (b - 2q \cos 2x)\, y = 0 \qquad (3,10)$$

in a Fourierexpansion of the following form:

$$y = \sum_{-\infty}^{\infty} B_h^{(n)} \sin hx. \qquad (3,11)$$

where the index n means that this solution for $x \to 0$ is near to n^2. (That is to say we will have $y = S_{e_n}(x)$).

The solution (3,11) is antimetrical. But we let fall now the requirement (3,9) and try to find the $B_h^{(n)}$ so that we have to compute only one butterfly-figure starting out of the point (n,0).

Therefore we put:

$$B_{n+2p} = \sum_{h=|p|}^{\infty} \beta_{n+2p,h} \, \tau^h \qquad (3,12)$$

(p can be positive and negative).
Again we have:

$$\beta_{n,0} = 1 \qquad \beta_{n,h} = 0 \text{ when } h > 0 \qquad (3,13)$$

and
$$b_n = \sum_{h=1}^{\infty} \alpha_h \tau^h + n^2 . \qquad (3,14)$$

As before we get the recursive formula:

$$\left\{ \sum_{h=1}^{\infty} \alpha_h \tau^h + 4\, nr - 4\, r^2 \right\} \sum_{h=|r|}^{\infty} \beta_{n-2r,h} \tau^h$$

$$- 4 \left\{ \sum_{h=|r+1|}^{\infty} \beta_{n-2r-2,h} \tau^{h+1} + \sum_{h= r-1}^{\infty} \beta_{n-2r+2,h} \tau^{h+1} = 0 \right. \qquad (3,15)$$

Put r = 0 and we find:

$$\alpha_h = 4 \left(\beta_{n-2,h-1} + \beta_{n+2,h-1} \right) . \qquad (3,16)$$

But now there is another complication: The first line of the butterfly is again expressed by:

$$\beta^{(n)}_{n-2p,p} = \frac{(n-p-1)!}{p!\,(n-1)!} \qquad (3,17)$$

That will say for $p = n$

$$\beta^{(n)}_{-n,n} = \infty .$$

When we wish to eliminate this difficulty, we have to choose a new coëfficiënt $\beta^{(n)}_{-n-2,n-1}$ and in order to satisfy our recurrence-relation (3,15) we have to build up our second butterfly starting from (-n,0). By this complication it is not easy to proof the convergence of the q-series of $B^{(n)}_m$.